THE
MATHEMATICAL GAZETTE.

EDITED BY
W. J. GREENSTREET, M.A.
WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc., AND PROF. E. T. WHITTAKER, M.A., F.R.S.
LONDON:
G. BELL AND SONS, LTD., PORTUGAL STREET, KINGSWAY,
AND BOMBAY.

VOL. IX. OCTOBER, 1919. No. 142.

THE GRAPHICAL TREATMENT OF DIFFERENTIAL EQUATIONS.

BY DR. S. BRODETSKY.

1. THE subject of this Paper is one that presented itself in a piece of work of a practical nature. The development of aeroplane flight naturally suggested the investigation of the motion of a body in a resisting medium. In general this problem was too difficult for solution. One therefore looked for legitimate means of simplifying it. One simple type of motion allied somewhat to aeroplane motion was that of a plane lamina moving in air under the earth's attraction. Even here difficulties arose in the solution of the resulting equations of motion; yet further simplification was not quite permissible, for care had to be taken that the simplified problem did bear some relation to the actual problem presented by nature. It was ultimately found that the simplest possible form of the problem gave rise to the differential equation

$$\frac{dy}{dx} = -\frac{x}{y} - (x^2+y^2)^{\frac{1}{2}},$$

or, putting

$$(x^2+y^2)^{\frac{1}{2}} = r,$$

$$\frac{dr}{dx} = -y.$$

The disconcerting feature about this equation was the fact that it was found to be quite insoluble by any of the "standard forms" dealt with in books on differential equations. Several mathematicians tried to discover a "transformation," or an "integrating factor," but without success. Yet a solution had to be found somehow.

It then occurred to the writer that where analysis had failed, geometry might succeed. The solution of a differential equation of the first order is of course represented geometrically by a family of curves. The ordinary treatment of differential equations consists in seeking for an analytical representation of these curves. Since, then, the analytical formula was apparently unobtainable, might not the curves themselves be "graphable"? The result was a complete success.

No claim is made for great originality. The graphical solution of differential equations is not a new idea. In particular, reference should be made to a

paper by Takeo Wada in *Memoirs of the College of Science, Kyoto Imperial University*, vol. ii., pp. 151-197, entitled "Graphical Solution of $dy/dx = f(x, y)$," which, although written after the present writer had developed his method of treatment, yet contains many references to older work on the subject. The object of the present paper is to present the method briefly and clearly, and especially to emphasise the pedagogical value of this method.

2. This is not the appropriate occasion for the discussion of the psychological problem whether there is a peculiar type of brain adapted to mathematical study. But it must be remarked that we very often fail in our pedagogical efforts, because we sometimes seem to demand a peculiar type of mathematical discipline. We cannot help insisting on mathematical neatness; but some-

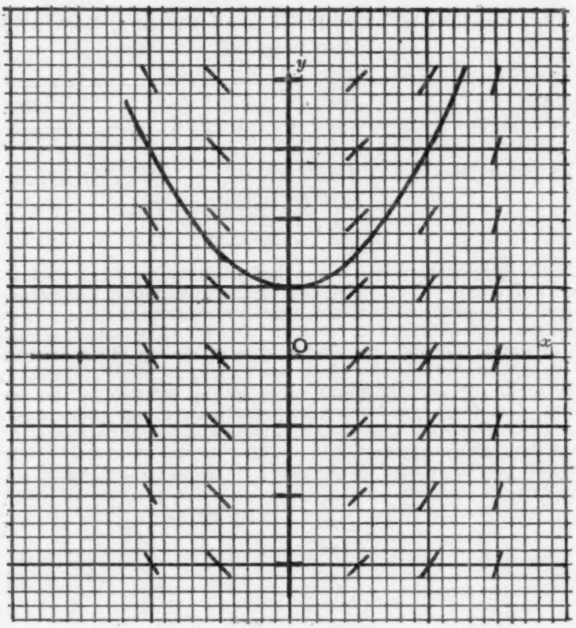


Fig. 1

times neatness degenerates into the perpetration of a series of tricks. And the result is that many pupils despair of ever learning the subject, because they think they are quite incapable of imitating what they suspect to be a species of jugglery.

In the case of differential equations, there is peculiar force in this suspicion. The usual school or college course in the subject consists of the "standard forms" and the treatment of "linear differential equations of the first order." In both branches the student is taught a series of tricks which he learns to apply, with more or less success, to such equations as are presented to him. There is a tacit understanding that in examinations only such equations will be set as are amenable, with comparative ease, to the set of tricks he has learnt. The result is that although the student may know theoretically that

every ordinary differential equation of the first order possesses a solution, in effect he is quite helpless when confronted by such an equation, unless he is solemnly assured that it can be reduced by some transformation to a type he can recognise as having learnt and memorised.

Modern progress in differential equations does not justify such a view. The researches in the past generation or two have taught us to discuss the properties of integrals of differential equations without any necessary reference to their integrability in the analytical sense. It would be in every way desirable that, in elementary courses on the subject, less emphasis should be laid on the integrable cases, and more on the information (of an algebraical or geometrical nature) that any equation as given affords about its solution.

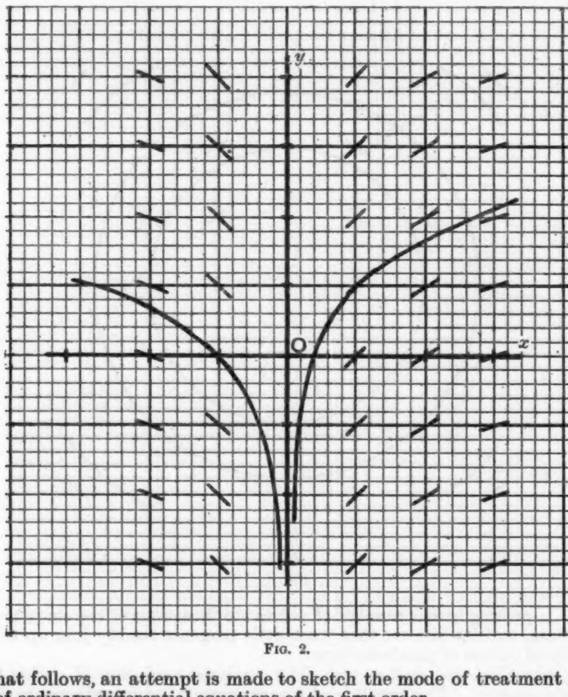


FIG. 2.

In what follows, an attempt is made to sketch the mode of treatment in the case of ordinary differential equations of the first order.

3. An ordinary differential equation of the first order is a relation between the differential coefficient of a function, the function itself and the argument of the function. Thus, if y is a function of x, such an equation tells us something about dy/dx, y and x. The problem is to find y. Now this problem can be presented in one of two forms. The usual one is: find an analytical expression for y in terms of x (or find each in terms of a parameter). But the geometrical form of the problem is as follows: given some information about the direction of the curve $y=y(x)$ at any point, find the curve. The latter form possesses one great advantage. Whereas the analytical problem can be solved in only a few special cases, the geometrical problem can be

answered with a certain degree of accuracy in all cases where the relation between dy/dx, y and x is not too complicated. In teaching, both forms of the problem should be presented and discussed.

The student should be taught that the differential equation gives us a geometrical property of a certain curve which has to be discovered. Thus $dy/dx=0$, not only means $y=$ *an arbitrary constant*, but also defines any straight line parallel to the axis of x. This follows geometrically from the fact that a curve defined by $dy/dx=0$ is everywhere parallel to the axis of x, and therefore must be a straight line parallel to the axis of x.

Again, consider the equation $dy/dx=x$. Its analytical solution is easy and readily interpreted geometrically. But let the student construct a figure as

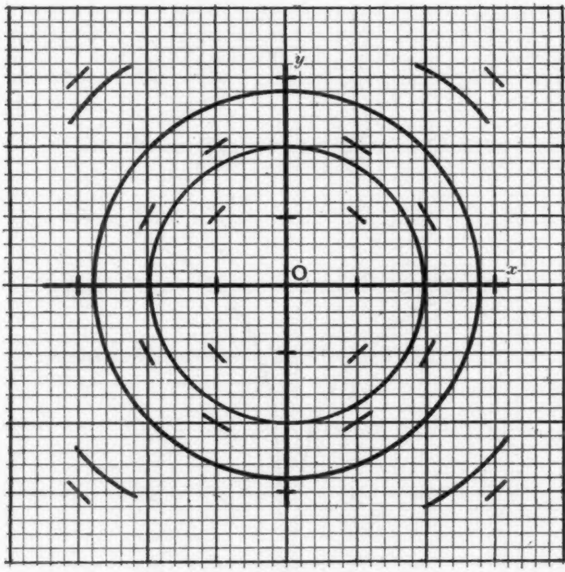


FIG. 3.

in Fig. 1, putting in at a number of points the directions of the curve through the points, and let him see that these little tangents readily join up to form a family of curves of parabolic form.

Or, take the case $dy/dx=1/x$. The analytical solution is $y=\log_e x+$ *an arbitrary constant*. This result is of course correct, but in ordinary pedagogy may be actually misleading. The elementary student is not likely to consider imaginary values of the arbitrary constant, and is thus liable to conclude that the solution only exists for positive values of x. But this is wrong. Let him draw Fig. 2, without reference to the analytical result. He will then get the logarithmic curves quite naturally, and will get them covering the whole plane.

Incidentally it is useful that the student should not lose sight of the fact that integral calculus is, in a sense, the full treatment of a particular type of differential equation. He should not be left to rediscover it by a mental shock.

Take the equation $dy/dx = -x/y$, which gives a family of concentric circles. If the direction is sketched at a number of points, as in Fig. 3, such a solution is suggested, which means that we get curves with different properties of convexity and concavity in different quadrants. The obvious process is therefore to find d^2y/dx^2. This can be done from the differential equation itself. Using the notation $dy/dx = y_1$, $d^2y/dx^2 = y_2$, we have

$$y_1 = -\frac{x}{y}, \quad y_2 = -\frac{1}{y} + \frac{x}{y^2}y_1 = -\frac{1}{y}\left(1 + \frac{x^2}{y^2}\right).$$

It follows that y_2 is negative for positive y and positive for negative y. Also y_1 is negative in the first and third quadrants, positive in the second and fourth

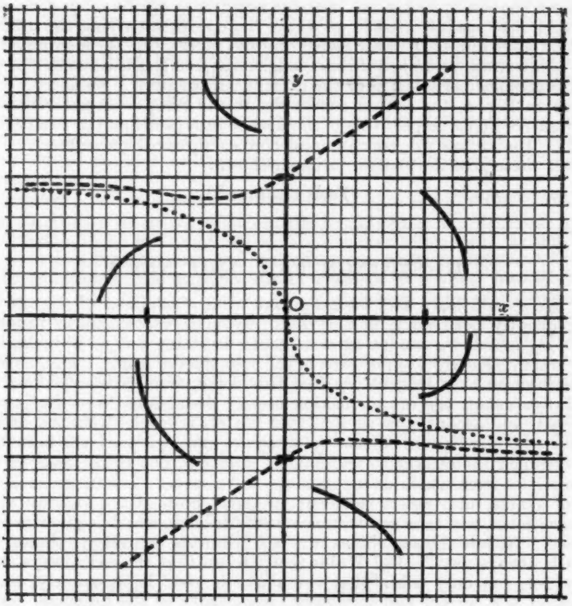


FIG. 4.

quadrants. We therefore divide up the x, y plane into four quadrants, and we know that the general shapes in these are

(1), (2), (3), (4).

The significance of the analytical solution (which happens to be obtainable) is now much more obvious and instructive.

4. It is true that the geometrical process is not essential for the cases we have considered so far. But it can be adopted whether an analytical process exists or not. Taking first the case where the differential equation is of the form

$$\frac{dy}{dx} = f(x, y).$$

where $f(x, y)$ is a definite function with no ambiguity, in other words *single valued*, we have the following process:

Draw the locus of all points at which the required family of curves are parallel to the axis of x: it is of course $f(x, y)=0$. Draw the locus of points where they are parallel to the axis of y, *i.e.* $\frac{1}{f(x, y)}=0$. One or other or both of these loci may not exist in the finite part of the plane; but in any case we get the plane divided up into a number of compartments: in some the required curves have positive dy/dx, in others negative dy/dx. Now calculate d^2y/dx^2 from the given differential equation. This can always be done. Draw the locus of points of inflexion, *i.e.* $d^2y/dx^2=0$. We now get a number of compartments, in some of which the curves are concave upwards, viz. d^2y/dx^2 positive, in others convex upwards, viz. d^2y/dx^2 negative. We have thus divided up the plane into spaces, in each of which the curves satisfying the differential equation have one of the four general forms (1), (2), (3), (4) mentioned above (§ 3). Now draw a number of short tangents at a convenient number of points, and the geometrical solution of the differential equation is obtained.

Thus, in Fig. 4, the dotted curves define the compartments with positive and negative dy/dx respectively; the dashed curves define the compartments in which the curves are concave and convex upwards respectively. Taken together these give the general forms as shown.

(*To be continued.*)

GLEANINGS FAR AND NEAR.

31. We should have said *logically*, but we are ashamed of the use which has frequently been made of this word, by mathematicians, in England at least. By *logical* we cannot agree to mean anything but an abbreviation of "that which is a correct application of the principles of logic"; and, on looking into writers on that subject, we find that logic, from Aristotle downwards, has always meant the art of making correct deductions from the principles employed, and accordingly we find that writers on logic, with the exception of a few who have imagined that metaphysics and logic were the same things, have confined themselves to methods of *deducing*, not to methods of *testing the principles* from which deductions are to be made. Let us go back to the time of Wallis, who was a sufficient specimen both of the logician and the mathematician, and take an example out of his book, which is given as correct *in logic*. "Where the sun shines it is day; but the sun always shines, therefore it is always day." Did Wallis really mean that the sun always shines? Surely not, but only this: that the above is good logic, namely, that the conclusion is a correct and necessary consequence of the premises, and that logic is simply the art of deducing correct and necesary deductions from premises. Now our books of controversial mathematics swarm with the use of the words *logical* and *illogical*, not as applied to methods of deducing, but as to the principles, from which deduction is to be made. One assumes infinitely small quantities, which is very *illogical*, says another; one approves of Euclid's axiom, which another says is against all good *logic*. It is clear then, that mathematicians must have got the habit, since the time they left off studying logic, of making the word *logical* stand for *right*, or *true*, or *reasonable*, or *proper*, or *correct*, or some such term. We therefore beg leave to use the term *correct* instead of logical, not that there would be any harm in making the word *logical* (or chemical) stand for *correct*, but only because, where there are two words meaning different things in etymology and usage out of mathematics, it is unnecessary to convert one into the other in them.—De Morgan, *Diff. and Integral Calculus*, 1836, n.p., 12 (*Introductory Chapter*).

CO-ORDINATE GEOMETRY IN SCHOOLS.*

BY W. J. DOBBS, M.A.

IN the Report of the Education Reform Council, published in 1917 under the title "Education Reform," it is suggested on page 83 that for boys of 14 to 16 years of age attending a secondary school it is desirable that the mathematical course should be directed towards an elementary use of calculus methods, so that those who remain at school to a later stage may be able to use effectively the elements of infinitesimal calculus. On the next page it is further suggested that boys of 16 to 18 years of age who take up mathematics and science may pursue the study of differential and integral calculus, including the easy parts of differential equations, and apply these with advantage to higher algebra, trigonometry, co-ordinate geometry, solid geometry and mechanics; that the co-ordinate geometry requires a much broader treatment than that commonly adopted—a treatment in which the calculus is freely used and in which some "geometrical" conics, when elegant and convenient, is dovetailed into the analysis.

The purpose of the present paper is to describe in somewhat greater detail a suitable course in co-ordinate geometry. Though some solid co-ordinate geometry should perhaps be included, I shall here confine my attention to plane geometry.

§ 1. **Gradient.** When a moving point referred to a pair of rectangular axes changes its position from (x, y) to (x_1, y_1), the gradient of the secant which passes through these two positions is $\Delta y/\Delta x = (y_1 - y)/(x_1 - x)$. If the path of the point is a straight line, and not otherwise, this gradient is constant. Thus the equation of the straight line of gradient m passing through (a, b) is $y - b = m(x - a)$. This we take as the fundamental form of the equation of a straight line and use it at the outset to the exclusion of all other forms. For instance, the join of $A(-1, 2)$ and $B(-3, -1)$ is of gradient $-3/(-2) = 3/2$. The gradient of every parallel to AB is also $3/2$. The equation of that parallel to AB which passes through $C(4, -3)$ is therefore $y + 3 = \frac{3}{2}(x - 4)$.

Now let A be a point whose co-ordinates are (h, k) with respect to a pair of rectangular axes OX, OY. Let OA be rotated forwards or backwards through a right angle into the position OB or OC. Then, clearly, the co-ordinates of B

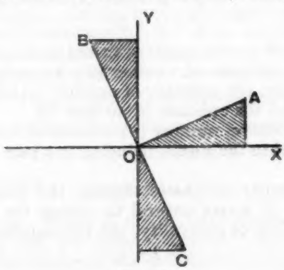


FIG. 1.

are $(-k, h)$ and the co-ordinates of C are $(k, -h)$. The gradient of OA is k/h; the gradient of OB is $h/(-k)$; the gradient of OC is $-h/k$. Thus BOC is a straight line of gradient $-h/k$. Now every parallel to OA is of gradient k/h,

* Read at a meeting of the London Branch of the Mathematical Association on 8th March, 1919.

and every perpendicular to OA has the same gradient as BOC, namely $-h/k$. Hence, the co-ordinate axes being at right angles, the product of the gradients of perpendicular lines is -1, and conversely.

Again, the distance from (h, k) up to the line whose equation is $Ax+By+C=0$, in the direction inclined at an angle θ to the x axis, is of magnitude r, given by

$$A(h+r\cos\theta)+B(k+r\sin\theta)+C=0;$$

whence

$$r=-\frac{Ah+Bk+C}{A\cos\theta+B\sin\theta}.$$

Now the gradient of the given line is $-A/B$, and therefore the gradient of every perpendicular to it is B/A. If then we require the perpendicular distance of (h, k) from the given line, we must choose θ so that $\tan\theta=B/A$, whence

$$A\cos\theta+B\sin\theta=\pm\sqrt{A^2+B^2},$$

$\therefore$ the perpendicular distance of the point (h, k) from the line $(Ax+By+C=0)$ is

$$\pm\frac{Ah+Bk+C}{\sqrt{A^2+B^2}}.$$

Proceeding on these lines, the beginner very quickly acquires sufficient command of the straight line to enable him to proceed to the treatment of curves.

§ 2. **Distance between Two Points.** Let (x_1, y_1) and (x_2, y_2) be the co-ordinates of any two points 1 and 2 with reference to a pair of rectangular axes. In passing from point 1 to point 2, $\Delta x=x_2-x_1$ and $\Delta y=y_2-y_1$. Now Δx and Δy may (either or both) be negative; but, whatever be their signs, their squares are positive. Hence, by Pythagoras' Theorem,

$$(\text{Distance-from-point-1-to-point-2})^2=(\Delta x)^2+(\Delta y)^2.$$

For instance, let (x, y) be the co-ordinates of a moving point which is tracing out a circle of radius c having its centre at (a, b). Then the distance of the tracing point from the centre is $\sqrt{(x-a)^2+(y-b)^2}$. Thus the equation of the circle is $(x-a)^2+(y-b)^2=c^2$ and is of the form $x^2+y^2=Ax+By+C$.

§ 3. **The Parabola.** Consider the two related curves

$$y=A+Bx, \quad\text{...(i)}$$

$$y^2=A+Bx. \quad\text{...(ii)}$$

For a common value of x, only positive values of the ordinate of number (i) give real values of the ordinate of number (ii), which is clearly symmetrical with respect to the x axis, the ordinate of number (ii) being the mean proportional between unity and the ordinate of number (i).

If B is zero and A positive, number (ii) consists of two straight lines equidistant from and parallel to the x axis, reducing to a pair of coincident straight lines when $A=0$.

If B is not zero, number (ii) passes through the points where number (i) meets $y=0$ and $y=1$. It seems natural to change the origin to the former point, so that, without loss of generality, the two equations become

$$y=Bx, \quad\text{...(i)}$$

$$y^2=Bx. \quad\text{...(ii)}$$

and in number (ii) only values of x which have the same sign as B give real values of y. It is easily seen that as we approach the origin the curve tends more and more to agree with $x=0$, and that in one direction of the x axis it opens out and extends to infinity.

Now consider the curve

$$y=A+Bx+Cx^2. \quad\text{...(iii)}$$

If C is not zero, we may write this equation in the form

$$y = A - B^2/4C + C(x + B/2C)^2.$$

Then, by moving the origin to a suitable point situated in the x axis, the equation is reduced to the form

$$y = A' + Cx^2.$$

There is therefore no loss of generality in taking $B = 0$ and writing the equation in the form

$$y = A + Cx^2. \qquad \text{(iii)}$$

By a further change of origin along the y axis the equation becomes

$$y = Cx^2,$$

$$\text{or} \quad x^2 = \frac{1}{C}y.$$

In this way the curve is identified with number (ii) considered above, the axis of symmetry being at right angles to its former direction.

§ 4. **The Central Conic.** Now consider the two related curves

$$y = A + Bx + Cx^2, \qquad \text{(iii)}$$

$$y^2 = A + Bx + Cx^2. \qquad \text{(iv)}$$

For a common value of x, only positive values of the ordinate of number (iii) give real values of the ordinate of number (iv), which is clearly symmetrical about the x axis, the ordinate of number (iv) being the mean proportional between unity and the ordinate of number (iii). Points common to number (iii) and the lines $y = 0$ and $y = 1$ are also situated on number (iv).

We will suppose that C is not zero, that case having been already considered. We have seen that number (iii) possesses an axis of symmetry parallel to the y axis. Hence number (iv) in all cases when C is not zero possesses a second axis of symmetry parallel to the y axis. It seems natural to change the origin to the point where this second axis of symmetry crosses the x axis, so that without loss of generality we may suppress B and take our equations in the form

$$y = A + Cx^2, \qquad \text{(iii)}$$

$$y^2 = A + Cx^2. \qquad \text{(iv)}$$

If A is zero and C positive, number (iv) consists of the two straight lines passing through the origin and the points common to $y = Cx^2$ and $y = 1$.

If A is not zero, the general appearance of the curves may be readily anticipated. Excluding the case in which A and C are both negative, as it gives no positive ordinate of number (iii), there are only three cases to consider—

(*a*) When A and C are both positive;

(*b*) When C is positive and A negative;

(*c*) When C is negative and A positive.

Now consider the circle

$$-Cy^2 = A + Cx^2 \qquad \text{(v)}$$

related to number (iv). When A and C have opposite signs, as in cases (*b*) and (*c*), this circle is real, its radius being $\sqrt{(-A/C)}$. Its centre is at the origin and it crosses the x axis at the same points as number (iv). It is called the auxiliary circle of number (iv).

In case (*c*) the two equations (iv) and (v) may be written thus:

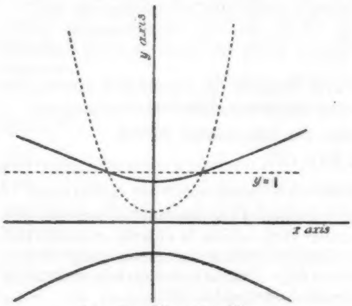
$$y = \sqrt{(-C)}\left\{\pm\sqrt{\frac{A + Cx^2}{-C}}\right\}, \qquad \text{(iv)}$$

$$y = \pm\sqrt{\frac{A + Cx^2}{-C}}. \qquad \text{(v)}$$

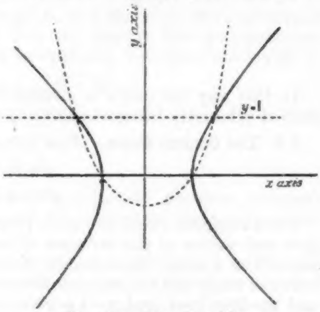
Thus it appears that number (iv) is a distorted circle, the distortion being made by multiplying every ordinate by $\surd(-C)$. When C lies between 0 and -1, the circle is contracted. When C lies between -1 and $-\infty$, the circle is expanded. If the circle whose equation is $x^2+y^2=a^2$ is contracted by multiplying every ordinate by b/a, b being less than a, the equation of the resulting curve is

$$x^2/a^2+y^2/b^2=1.$$

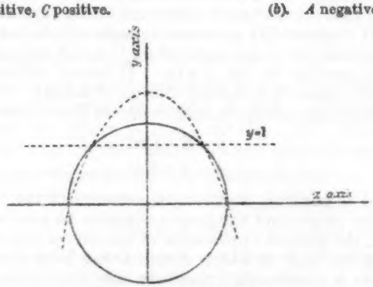
This distortion is exactly that which takes place when a circle is orthogonally projected upon a plane inclined to its own plane at an angle $\cos^{-1}(b/a)$, and certain geometrical properties of the circle remain unchanged by projection.



(a). A positive, C positive.



(b). A negative, C positive.



(c). A positive, C negative.

FIG. 2.

If the circle whose equation is $x^2+y^2=b^2$ is expanded by multiplying every ordinate by a/b, a being greater than b, the equation of the resulting curve is $x^2/b^2+y^2/a^2=1$—the same curve as before with the axes interchanged.

In case (b) the connection between the conic and its auxiliary circle is not so apparent, and appears to suggest an imaginary coefficient of distortion.

In case (a) no auxiliary circle has yet appeared; but, writing the equation in the form

$$x^2=-\frac{A}{C}+\frac{1}{C}y^2,$$

and remembering that A and C are both positive, case (a) is by an interchange of axes identified with case (b).

The beginner may gain familiarity with the general appearance of a hyperbola in the following manner: Consider first the simple case $y^2=a^2+x^2$. Take the fixed point A at $(0, a)$ and the variable point N at $(x, 0)$. Then draw

the ordinate $NP = \pm NA$. The locus of P gives the curve. The equation $y^2 = x^2 - a^2$ represents the same curve with the axes interchanged. Both curves are easily identified with $2xy = a^2$ by changing the axes of coordinates to the bisectors of the angles between the axes of symmetry.

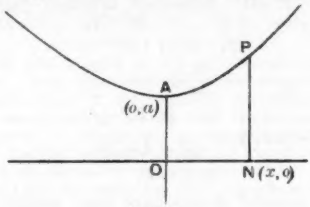


FIG. 3.

If the curve $x^2 - y^2 = a^2$ is distorted by multiplying every ordinate by b/a, the equation of the resulting curve is $x^2/a^2 - y^2/b^2 = 1$.

It appears that the parabola has one axis of symmetry only, meeting the curve at one real point only, called its vertex; the ellipse has two axes of symmetry, a major and a minor axis, meeting the curve in four real points, the extremities of the major axis being called the vertices of the ellipse; the hyperbola has two axes of symmetry, one of which only meets the curve in two real points called its vertices.

Taking a vertex as origin and an axis of symmetry as x axis, all cases are included in the single equation

$$y^2 = Bx + Cx^2. \qquad \text{(vi)}$$

There is no loss of generality in assuming that B is positive, since the reversal of the x axis would change $Bx + Cx^2$ into $-Bx + Cx^2$. If C is zero, number (vi) is a parabola. If C is not zero, the equation of the auxiliary circle is

$$-Cy^2 = Bx + Cx^2. \qquad \text{(vii)}$$

The coefficient of distortion is therefore $\sqrt{(-C)}$.

Number (vi) is a hyperbola if C is positive, an ellipse if C is negative, and, the x axis being the major axis, C, when negative, lies between 0 and -1, *i.e.* $C+1$ is positive. When $C = -1$, numbers (vi) and (vii) are identical. The centre of number (vii) is at $(-B/2C, 0)$; its radius is $B/2C$ if B and C are both positive, $-B/2C$ if B is positive and C negative.

§ 5. **Chord Properties.** Consider the curve

$$y = A + Bx + Cx^2, \qquad \text{(iii)}$$

and let the tracing point travel along the curve from (x, y) to (x_1, y_1).

Then
$$\begin{cases} y = A + Bx + Cx^2 \\ y_1 = A + Bx_1 + Cx_1^2 \end{cases};$$

$$\therefore\ y_1 - y = B(x_1 - x) + C(x_1 + x)(x_1 - x);$$

$$\therefore\ \frac{\Delta y}{\Delta x} = B + C(x_1 + x).$$

Denoting the middle point of the chord by (X, Y) and its gradient by m, we have

$$m = B + 2CX.$$

Thus parallel chords have their middle points situated in a parallel to the axis of symmetry.

Now consider the two related curves

$$\begin{cases} y = f(x) \\ y^2 = f(x) \end{cases}.$$

In order to distinguish between the two, we will use (x, η) to denote a point on the former and μ the gradient of a chord, (x, y) to denote a corresponding point on the latter and m the gradient of the corresponding chord.

Thus $$y^2 = \eta$$
and $$y_1^2 = \eta_1;$$
$$\therefore \ (y_1 - y)(y_1 + y) = \eta_1 - \eta;$$
$$\therefore \ \frac{\Delta y}{\Delta x}(y_1 + y) = \frac{\Delta \eta}{\Delta x},$$
$$i.e. \quad m \cdot 2Y = \mu.$$

For instance, in the parabola
$$y^2 = A + Bx,$$
$$2mY = B,$$
howing again that parallel chords have their middle points situated in a parallel to the axis of symmetry.

Again, in the central conic $y^2 = A + Cx^2$,
$$2mY = 2CX;$$
$$\therefore \ Y = \frac{C}{m} \cdot X,$$
showing that conjugate diameters have the product of their gradients equal to C.

§ 6. **Gradients of Curves.** Tangent properties may be deduced from chord properties. The student should, however, by this time be able to use the differential calculus directly. Thus,
$$y^2 = A + Bx + Cx^2;$$
$$\therefore \ 2y\frac{dy}{dx} = B + 2Cx; \qquad \therefore \ \frac{dy}{dx} = \frac{B + 2Cx}{2y};$$
$\therefore$ the gradient of the tangent at (x_1, y_1) is
$$\frac{B + 2Cx_1}{2y_1};$$
$\therefore$ the equation of the tangent at (x_1, y_1) is
$$y - y_1 = \frac{B + 2Cx_1}{2y_1}(x - x_1),$$
which transforms into
$$2y_1 y = 2A + B(x + x_1) + 2Cx_1 x.$$

Now consider the curve
$$y^2 = f(x).$$
We have
$$2y\frac{dy}{dx} = f'(x),$$
i.e. the subnormal is half the gradient of the related curve $y = f(x)$.

Also
$$y \Big/ \frac{dy}{dx} = y^2/\tfrac{1}{2}f'(x) = 2f(x)/f'(x),$$
i.e. the subtangent is twice the subtangent of the related curve $y = f(x)$.

For instance, in the parabola $y^2 = A + Bx$, the subnormal is constant, being equal to $\frac{1}{2}B$, and the part of the tangent between the axis and the point of contact is bisected by the tangent at the vertex.

Again, in the central conic $y^2 = A + Cx^2$, the subnormal is Cx, and the subtangent is $x + A/Cx$, so that the part of the x axis intercepted between the origin and the tangent is $-A/Cx$.

(To be continued.)

REVIEWS.

(i) Analytic Geometry and Calculus. By F. S. Woods and F. H. Bailey. Pp. xi+516. 10s. 6d. net. 1918. (Ginn and Co.)

(ii) Differential and Integral Calculus. By H. B. Phillips, Ph.D. Pp. 162+194. $2.00, or in two volumes, $1.25 each. (Wiley and Sons; Chapman and Hall.)

(i) This volume is an abridgment of the well-known course in Mathematics by the same authors. They have omitted those subjects which seem more suited to the specialist than to the engineer or ordinary student, such as general theorems in analytical geometry and determinants, and they have curtailed their treatment of differential equations and special types of integration: some new matter includes empirical equations and approximate integration. There is an abundant supply of examples, mainly of a straightforward character, and the diagrams are clear, the most interesting being one which illustrates successive approximations to a Fourier Series.

(ii) This text-book aims at giving the student a thorough practical knowledge of calculus ideas, and is based on intuitional methods. It differs from the introductory type of book now in favour in this country, in that it makes fairly large demands on the manipulative power of the reader, or at any rate furnishes him with many opportunities for developing that power; it is perhaps surprising that under such circumstances the author avoids using hyperbolic functions in his chapter on general integration.

Empirical Formulas. By T R. Running. Pp. 143. 7s. 1919. (Chapman and Hall.)

The author's object is to enable the engineer who has at his disposal a group of data derived from experiment to calculate an approximate formula which will summarise the results within the limits investigated. Formulae so obtained will not usually correspond to or represent any underlying physical law, but will serve to indicate what may be expected to occur within confined limits, which is all that the practical engineer is likely to require. The method of differences is explained at some length, and applications are made to simple functions of polynomials. Other functions dealt with are a^x, x^a, $e^{ax}\sin bx$ and Fourier Series, in the last-named calculations being based on Runge's 12-ordinate method. There is a short account of the Principle of Least Squares and a useful chapter on Interpolation. We regard this monograph as a valuable contribution to practical work. C. V. D.

Annuaire pour l'An 1919. Publié par le Bureau des Longitudes. 3 frs. 1919. (Gauthier-Villars.)

The *Annuaire* has appeared with the regularity of clock-work since 1870, but on the present occasion—with a change in accordance with our general experience of the times in which we live—its price is doubled. It is still among the cheapest books to be bought. This year we have 326 pp. with the usual "calendrier," and astronomical tables and tables of weights and measures. The summary entitled "Solar Physics," pp. 228-244, is by M. H. Deslandres. This year it is the turn of geographical and demographical tables, and tables of interest, annuities, etc., the former having reference in the main to France and her colonies. These take up about 200 pp.

The essays, which are always a notable feature in the *Annuaire*, are this year by MM. Appell and Hamy. The former finds a congenial topic in "The figures of relative equilibrium of a rotating homogeneous liquid" (pp. 60). After an historical sketch of the theory up to the work of the late MM. Liapounoff and Poincaré, he deals with the problem of stability and the phenomena at the point of bifurcation. The value of the paper is greatly enhanced by a full bibliography. M. Hamy treats of the inference of the real diameters of minor planets and satellites from a study of the interference fringes (pp. 27). He believes that, with the more powerful instrumental opportunities at our disposal in the 100-in. reflector at Mount Wilson, it will be quite possible to determine the angular-diameters of 1st magnitude stars. The excellent table of contents runs to 69 pp.

Theory of Maxima and Minima. By H. Hancock. Pp. 189. 10s. 6d. net. 1917. (Ginn & Co.)

This is a carefully written account of the theory of maxima and minima; the "ambiguous case" and prevalent errors concerning it are dealt with at some length. The theory for functions of several variables, both when independent and when subject to equations of condition, is treated with less elaboration. We are then shown when we can avoid the laborious methods which precede, and there is tacked on a disconnected chapter on general theory of functions.

Geometrical ideas are used timidly, even in geometrical applications; the examples given are often so simple that quite elementary methods would have served, and the use of the heavier theory is not illustrated. Where delta and epsilon reign supreme, we must not expect charm of manner, but there is attraction in the writer's enthusiasm for Weierstrass and the other mathematicians from whom he has learnt his subject.

Where accuracy is so essential it is disappointing to find a slip in one of the earliest definitions on p. 2, which is, however, avoided in the corresponding passage on p. 19. H. P. H.

Scritti Mathematici Offerti ad Enrico d'Ovidio. Pp. 384. Lire 30. 1918. (Fr. Bocca, Turin.)

It is a pretty Italian custom to mark an epoch in the life of a great scholar, by presenting to him a set of scientific notes written by his friends in his honour. On the occasion of the retirement of Prof. d'Ovidio from the chair at Turin, at the age of 75 and after 46 years' service, the present volume is contributed by a distinguished company of nine former assistants, ten old students and one other friend of the veteran.

The plan has its drawbacks from the point of view of the students who come after, as such a book is apt to become difficult of access and the papers to escape notice; so it is quite suitable that most of them deal with generalisations or more elegant proofs of known results, rather than with new theory. The longest paper, F. Gerbaldi on continued fractions, is one of a series of three, the other two appearing in different periodicals; the shortest is G. Peano's three pages on an interpolation formula. A controversial note is struck in B. Levi's statement that Zermelo's postulate is entirely meaningless. To a geometer the most attractive item is A. Pensa's generalisation of d'Ocagne's transformation, bringing out new relations between many familiar plane curves. The subjects of the other papers range from reciprocation in n dimensions to radiotelegraphy. H. P. H.

Differential Equations. By H. Bateman. (Longmans' Mathematical Series.) Pp. xi+306. 16s. net. 1918. (Longmans.)

The author shows in this treatise a wide range of reading and careful study of his subject; and indeed not only of the theory of Differential Equations, but of a vast field of mathematical investigation for which a knowledge of the theory is essential, especially in the domain of Mathematical Physics. References are very plentiful, and the reader will find the book of great assistance in putting him in touch with the very numerous researches which have been undertaken in this favourite field of study.

The contents include the method of solution of ordinary and total differential equations, partial equations of the first and second orders, integration in series, solutions by means of definite integrals, and mechanical integration, with applications to many branches of applied mathematics. The theory of differential equations from the side of Continuous Groups and Theory of Functions is not included except incidentally.

It is doubtful whether the book could be used with advantage by a beginner. The arrangement of the material is not such as to make the book easy reading, while simple and intricate reasoning alternate in a very confusing manner. The examples are numerous, but absence of solutions is in this subject a very serious drawback to the young student. The idea of leading up to the pure theory by means of applications in applied mathematics has decided advantages, and probably follows the historical order of development. But the

notion has not been followed out with complete success. The examples are drawn in the main from branches of Mathematical Physics which must be quite unintelligible to anyone who has not a pretty thorough acquaintance with all that this book is supposed to teach him; and, quite apart from that, they are in the main far too difficult. Perhaps a larger number of easy geometrical illustrations and fewer applications from Physics would have been advisable.

The solution of the interesting problem of the decomposition of radioactive substances in § 14 seems to be incorrect (cf. Hilton's *Linear Substitutions*, p. 88); for without the limitation $\lambda_0 > \lambda_1 > \lambda_2 > \ldots$ we might have P_m negative. In practice these inequalities are not satisfied. We must appeal to the physicists for the explanation of this very interesting paradox.

H. H.

CORRESPONDENCE.

To the Editor of the *Mathematical Gazette*.

"PIARAR."

Sir,—On the covers of certain answer books given out to candidates for Government examinations, the area of a circle is stated to be πrr.

It would be interesting to learn what precedent the framers of these books had for the use of a double letter to denote what most sensible people call a square and represent by r^2.

Certainly arithmetic affords no justification for the innovation, for 77 means seventy-seven, and when it is necessary to represent the square of 7 the pupil is taught the better notation 7^2.

Neither is the change justified by algebra. It is true that in beginning this subject, pupils are taught that when two letters are placed together like *ab* with no explanatory sign, they are to be multiplied and not added; but as they have already learnt the notation for squares, there is no need to introduce them to an inconvenient and unfamiliar substitute for a recognised notation.

I have seen other books in which the formula is given in the more explicit form $\pi \times r \times r$. To this the objection does not apply, but if anything is required which is intermediate between this and the standard notation πr^2 the proper choice to make is $\pi \times r^2$ or even $\pi r \times r$, certainly not πrr.

I am, Sir, yours faithfully,

"Piexarsquared."

"TEN, TWELVE, OR SIXTY."

Sir,—At present we are being flooded with appeals for the introduction of a decimal system of weights, measures and coinage. This has been met by an unanswerable objection in a letter in the *Morning Post* pointing out the inconvenience of the number ten, and urging the teaching of a duodecimal scale of notation.

Of course this is the proper solution of the whole question. But the difficulty remains that, in any problem, we have to take account of human inertia, ignorance, prejudice and obstinacy. The first thing we shall be told is that "ten is a much more convenient number than twelve, because in multiplying or dividing by ten you only have to put on or take off a 0 at the end."

Now there is one plan which meets this objection whatever be its disadvantages in other respects.

When the Metric System was introduced, the introducers stopped short of dividing the day into 25 hours or 100 quarters, each large interval being divided into 10 or possibly 100 minutes and each minute into 100 seconds. On the contrary, the factor 60 was retained for minutes and seconds, and everybody of whatever nationality uses it.

It would thus be quite easy to introduce systems of weights, measures and coinage based on the factor 60. As applied, for example, to coinage, the smallest coin commonly used in France would be denoted by 3 units instead of 5 centimes, and it would thus be easy to purchase one-third of a franc's worth of goods, which is now impossible. In England one penny might be divided into five instead of four farthings, 60 pence would make a crown, and one pound would be one-third of the next higher unit. In weights and measures, we could obtain units of the dimensions of half an ounce, a kilogram, a hundredweight and three tons; again, half an inch, a yard and two miles might be the rough equivalents of the units of length. It would certainly be easier to change to a sexagesimal system of weights and measures from a decimal system, or from our own system, and we might well set the fashion with our coinage. G. H. Bryan.

University Observatory,
Oxford, July 6, 1919.

Dear Sir,—I was on the point of writing to you to express joy when Professor Bryan published his Bordered Antilogarithm Table in December 1915; his extended tables in the *Gazette* for May afford an even better opportunity. The tables are all good; if I am specially glad to see the table of exact squares it is no disparagement to the others. My selfish reason will be found in *Monthly Notices R.A.S.* lxxv. p. 530 (1915, May). Perhaps I may repeat here the device for using a table of *exact* squares to three figures to get squares of six figures. The theorem is

$$\begin{aligned}(x+10^{-3}y)^2 = x^2 &+ 10^{-3}\,.\,x^2 \\ &+ 10^{-3}\,.\,y^2 + 30^{-6}\,.\,y^2 \\ &- 10^{-3}(x-y)^2.\end{aligned}$$

Thus the square of 123456 is written down as follows by the help of Professor Bryan's brief table:

$$\begin{array}{rr} 123^2 = & 15129 \\ 10^{-3}\times 123^2 = & 15\ 129 \\ 10^{-3}\times 456^2 = & 207\ 936 \\ 10^{-6}\times 456^2 = & 207\ 936 \\ \hline -10^{-3}\times 333^2 = - & 110\ 889 \\ \hline 123456^2 = & 15241\ 383\ 936 \end{array}$$

—Yours truly, H. H. Turner.

P. E. B. JOURDAIN, M.A.

OCT. 1, 1919

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GLASGOW: PRINTED AT THE UNIVERSITY PRESS BY ROBERT MACLEHOSE AND CO. LTD.